Infinitesimal spectral rigidity of symmetric spaces (Lie Groups, Geometric Structures and Differential Equations : One Hundred Years after Sophus Lie)

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数理解析研究所講究録
1150 巻 2000 年 96-104

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1. Main results

We are reporting on joint work with Jacques Gasqui; some of these results were announced in [10]. We are interested in determining which irreducible symmetric spaces of compact type are infinitesimally spectrally rigid (i.e., spectrally rigid to first order).

Let (X, g) be a Riemannian symmetric space of compact type. Consider a family of Riemannian metrics $\{g_t\}$ on X, with $g_0 = g$. We say that $\{g_t\}$ is an isospectral deformation of g if the spectrum of the Laplacian of the metric g_t is independent of t.

In [12], V. Guillemin proves that the infinitesimal deformation $h = \frac{d}{dt} g_{t|t=0}$ of an isospectral deformation $\{g_t\}$ of g satisfies the following integral condition: for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z h(\zeta, \zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z. If all of these integrals corresponding to a symmetric 2-form h on X vanish, we say that h satisfies the Guillemin condition.

If a deformation $\{g_t\}$ of g is trivial, that is, if there exists a family of diffeomorphisms $\{\varphi_t\}$ of X such that $\varphi_t^* g_t = g$, then the infinitesimal deformation $\frac{d}{dt} g_{t|t=0}$ of $\{g_t\}$ is a Lie derivative of the metric. Such Lie derivatives always satisfy the Guillemin condition. We are led to the following:

DEFINITION. *We say that the space (X, g) is rigid in the sense of Guillemin if the only symmetric 2-forms on X satisfying the Guillemin condition are the Lie derivatives of the metric g.*

Guillemin's result gives us a criterion for infinitesimal spectral rigidity which may be restated as follows:

THEOREM 1. *If the symmetric space X is rigid in the sense of Guillemin, it is infinitesimally spectrally rigid.*

Spheres are not rigid in the sense of Guillemin. The Guillemin rigidity of the spaces of compact type and of rank one (i.e. the projective spaces) which are not spheres was proved by Michel [15] for the real projective space $\mathbb{RP}^n$, with $n \geq 2$, and by Michel [15] and Tsukamoto [16] for the other projective spaces (see also [3], [6] and [7]).

The author was supported in part by NSF Grant DMS 96-25633.

Let $\mathbb{K}$ be a division algebra over $\mathbb{R}$ (i.e. $\mathbb{K}$ is equal to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) and let $m, n \geq 1$ be given integers. The Grassmannian $G^{\mathbb{K}}_{m,n}$ of all $\mathbb{K}$-planes of dimension m in $\mathbb{K}^{m+n}$ is an irreducible symmetric space of compact type whose rank is $\min(m,n)$, with the exception of $G^{\mathbb{R}}_{1,1} = S^1$ and $G^{\mathbb{R}}_{2,2}$; the universal covering space of $G^{\mathbb{R}}_{2,2}$ is $S^2 \times S^2$.

Our main result may be stated as follows:

THEOREM 2. *For $m, n \geq 2$, with $m \neq n$, the Grassmannian $G^{\mathbb{K}}_{m,n}$ is rigid in the sense of Guillemin.*

This implies that the Grassmannian $G^{\mathbb{K}}_{m,n}$, with $m, n \geq 2$ and $m \neq n$, is infinitesimally spectrally rigid and provides us with the first examples of irreducible symmetric spaces of arbitrary rank having this property. Theorem 2 together with the results of Michel–Tsukamoto show that a Grassmannian, which is an irreducible symmetric space of compact type and which is equal to its adjoint space, is rigid in the sense of Guillemin and so is infinitesimally spectrally rigid.

2. The maximal flat Radon transform

Let (X, g) be a symmetric space of compact type, whose tangent and cotangent bundles we denote T and T^*, respectively. We consider the p-th symmetric product S^pT^* and the j-th exterior product $\bigwedge^j T^*$ of T^*. If E is a vector bundle over X, we denote by $E_{\mathbb{C}}$ its complexification and by $C^\infty(E)$ the space of global sections of E over X. We may write X as a homogeneous space G/K, where G is a compact connected semi-simple Lie group, which acts on X by isometries, and K is the isotropy subgroup of G at a point of X; we may suppose that (G, K) is a Riemannian symmetric pair.

The space Ξ of all maximal flat totally geodesic tori of X is a homogeneous space of G. The space $C^\infty(X)$ (resp. $C^\infty(\Xi)$) of all real-valued functions on X (resp. on Ξ) is a G-module. The maximal flat Radon transform of X studied by Grinberg [11] is the G-equivariant linear mapping

$$I : C^\infty(X) \to C^\infty(\Xi),$$

which assigns to a function f on X the function $\hat{f}$ on Ξ, whose value at a torus $Z \in \Xi$ is the integral of f over Z.

In [10], we define a maximal flat Radon transform I_p for symmetric p-forms which assigns to a symmetric p-form on X a section of a vector bundle over Ξ (which depends only on p). On functions, I_0 coincides with the mapping I considered by Grinberg. The space $\mathcal{N}$ of all symmetric 2-forms on X satisfying the Guillemin condition is the G-submodule of $C^\infty(S^2T^*)$ equal to the kernel of the maximal flat Radon transform I_2. The space X is rigid in the sense of Guillemin if and only if

$$\{ \mathcal{L}_\xi g \mid \xi \in C^\infty(T) \} = \mathcal{N}.$$

The space (X, g) is an Einstein manifold; in fact, the metric g satisfies

$$\mathrm{Ric}(g) = \lambda g,$$

where $\lambda > 0$. Using the theorems of Lichnerowicz and Obata concerning the first non-zero eigenvalue of the Laplacian of a compact Einstein manifold with positive Ricci curvature (see [2]), we prove:

PROPOSITION 1. *Let X be an irreducible symmetric space of compact type, which is not isometric to a sphere. If X is rigid in the sense of Guillemin, then the maximal flat Radon transform (for functions) on X is injective.*

We recall that the adjoint space of X is the symmetric space which admits X as a Riemannian cover and which is itself not a Riemannian cover of another symmetric space.

EXAMPLES:

1) The adjoint space of the n-sphere S^n is the projective space $\mathbb{RP}^n$. For these spaces of rank one, the maximal flat tori are the closed geodesics (i.e. the great circles). A function on $\mathbb{RP}^n$ lifts to an even function on S^n, and all the even functions on S^n arise in this manner. The kernel of the maximal flat Radon transform for functions on S^n is the space of all odd functions on S^n. In fact, this Radon transform is injective when restricted to the even functions on S^n; this is equivalent to the classic fact that the Radon transform for functions on $\mathbb{RP}^n$ is injective.

2) The adjoint space of the Grassmannian of oriented m-planes in $\mathbb{R}^{m+n}$ is equal to $G^{\mathbb{R}}_{m,n}$, when $m \neq n$.

3) When $m, n \geq 2$ and $m \neq n$, the Grassmannian $G^{\mathbb{K}}_{m,n}$ is equal to its adjoint space. The Grassmannian $G^{\mathbb{K}}_{1,n}$ is the projective space $\mathbb{KP}^n$ and, when $n \geq 2$, it is equal to its adjoint space.

In [11], Grinberg generalizes the results concerning the maximal flat Radon transform for functions on S^n and $\mathbb{RP}^n$ and proves:

THEOREM 3. *The maximal flat Radon transform for functions on X is injective if and only if the space X is equal to its adjoint space.*

Since the sphere S^n is not rigid in the sense of Guillemin, Proposition 1 and Theorem 3 gives the following necessary condition for Guillemin rigidity:

THEOREM 4. *Let X be an irreducible symmetric space of compact type. If X is rigid in the sense of Guillemin, then X is equal to its adjoint space.*

3. First method: harmonic analysis

We consider the isotypic component $C^\infty_\gamma(F)$ of a complex homogeneous vector bundle F over the homogeneous space X corresponding to an element γ of the dual $\hat{G}$ of the group G.

The Killing operator $D_0 : C^\infty(T) \to C^\infty(S^2T^*)$, sending $\xi \in C^\infty(T)$ into $\mathcal{L}_\xi g$, is homogeneous, and so

$$D_0 C^\infty_\gamma(T_{\mathbb{C}}) \subset C^\infty_\gamma(S^2T^*_{\mathbb{C}}).$$

We view the complexification $\mathcal{N}_{\mathbb{C}}$ of the space $\mathcal{N}$ as a G-submodule of $C^\infty(S^2T^*_{\mathbb{C}})$; it consists of all complex symmetric 2-forms on X satisfying the Guillemin condition.

From the fact that the direct sum

$$\bigoplus_{\gamma\in\Gamma} C^\infty_\gamma(S^2T^*_{\mathbb{C}})$$

is a dense subspace of $C^\infty(S^2T^*_{\mathbb{C}})$, we infer that

PROPOSITION 2. *The space (X,g) is rigid in the sense of Guillemin if and only if*

$$\mathcal{N}_{\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}}) = D_0 C^\infty_\gamma(T_{\mathbb{C}}), \tag{1}$$

for all $\gamma \in \hat{G}$.

To prove the Guillemin rigidity of X, it is sufficient to:

(i) For all $\gamma \in \hat{G}$, determine the multiplicities of the G-modules $C^\infty_\gamma(T_{\mathbb{C}})$ and $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$.

(ii) Describe an explicit basis for the space of highest weight vectors of the G-module $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$, for $\gamma \in \hat{G}$.

(iii) Consider the action of the Radon transform on these basis vectors to prove the equality (1) for all $\gamma \in \hat{G}$.

In [9], we used these methods to prove:

THEOREM 5. *The real Grassmannian $G^{\mathbb{R}}_{2,3}$ is rigid in the sense of Guillemin.*

Because all the Grassmannians $G^{\mathbb{R}}_{2,n}$, with $n \geq 3$, are of rank 2, this theorem implies the real Grassmannian $G^{\mathbb{R}}_{2,n}$ is rigid in the sense of Guillemin, for all $n \geq 3$.

4. Differential operators and Einstein deformations

Let B be the sub-bundle of $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$ of all curvature-like tensors on X. We consider the natural trace mappings

$$\mathrm{Tr} = S^2T^* \to \mathbb{R}, \qquad \mathrm{Tr} : \textstyle\bigwedge^2 T^* \otimes \bigwedge^2 T^* \to S^2T^*.$$

Let $S^2_0T^*$ be the sub-bundle of S^2T^* equal to the kernel of $\mathrm{Tr} : S^2T^* \to \mathbb{R}$. It is easily seen that

$$\mathrm{Tr}\, B \subset S^2T^*.$$

The infinitesimal orbit of the curvature

$$\tilde{B} = \{\, \rho(u) \mid u \in T^* \otimes T,\ \rho(u)g = 0 \,\}$$

is a sub-bundle of B. In [5], we constructed an explicit second-order differential operator

$$D_1 : C^\infty(S^2T^*) \to C^\infty(B/\tilde{B}),$$

which is part of the compatibility condition of the Killing operator D_0. Thus we obtain a complex

$$C^\infty(T) \xrightarrow{D_0} C^\infty(S^2T^*) \xrightarrow{D_1} C^\infty(B/\tilde{B}). \tag{2}$$

Since the operator D_0 is elliptic, the cohomology of this complex is isomorphic to the space

$$H(X) = \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0,\ D_1 h = 0 \,\}.$$

If the space X has constant curvature, then we have $\tilde{B} = \{0\}$, and the sequence (2) is the one introduced by Calabi [4] and is *exact* (see also [5]).

Since (X, g) is an Einstein manifold and $\mathrm{Ric} = \lambda g$, we know that $\operatorname{Tr} \tilde{B} = \{0\}$. Thus the mapping Tr induces is a well-defined trace mapping

$$\operatorname{Tr} : B/\tilde{B} \to S^2T^*.$$

The divergence $\operatorname{div} h$ of a symmetric 2-form h on X is a section of T^*. If f is a real-valued function on X, we denote by $\operatorname{Hess} f$ the Hessian of f. On the Einstein manifold X, the differential operator D_1 is related to the Lichnerowicz Laplacian

$$\Delta : C^\infty(S^2T^*) \to C^\infty(S^2T^*)$$

acting on symmetric 2-forms; in fact, if h is an element of $C^\infty(S^2T^*)$ satisfying $\operatorname{div} h = 0$, we have

$$\operatorname{Tr} D_1 h = -\tfrac{1}{2}(\Delta h - \operatorname{Hess} \operatorname{Tr} h) \tag{3}$$

(see [8]). By means of Lichnerowicz's Theorem concerning the first non-zero eigenvalue of the Laplacian (acting on complex-valued functions) of a compact Einstein manifolds with positive Ricci curvature (see [2]), from the relation (3) we deduce the following:

LEMMA 1. *Let N be a sub-bundle of B containing $\tilde{B}$ and E be a sub-bundle of $S^2_0T^*$ satisfying $\operatorname{Tr} N \subset E$. Let h be an element of $C^\infty(S^2T^*)$ satisfying*

$$\operatorname{div} h = 0, \qquad D_1 h \in C^\infty(N/\tilde{B}).$$

Then we have

$$\operatorname{Tr} h = 0, \qquad \Delta h - 2\lambda h \in C^\infty(E).$$

In [1], Berger and Ebin introduced the (finite-dimensional) space of infinitesimal Einstein deformations

$$E(X) = \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0,\ \operatorname{Tr} h = 0,\ \Delta h = 2\lambda h \,\}$$

of the metric g (see also [13]).

If we take $N = \tilde{B}$ and $E = \{0\}$ in Lemma 1, we obtain the following:

LEMMA 2. *The space $H(X)$ is finite-dimensional and is a subspace of $E(X)$.*

THEOREM 6. *Let X be an irreducible symmetric space of compact type. If $E(X) = \{0\}$, then the sequence (2) is exact.*

Let $\mathfrak{g}$ be complexification of the Lie algebra of G. If X is not equal to a simple Lie group, then $\mathfrak{g}$ is an irreducible G-module. According to Koiso [13], the Lichnerowicz Laplacian Δ is equal to the Casimir operator of the G-module $C^\infty(S^2T^*_{\mathbb{C}})$. From this fact, we obtain:

PROPOSITION 3. *Suppose that X is not equal to a simple Lie group. Let γ_0 be the element of $\hat{G}$ which is the equivalence class of the irreducible G-module $\mathfrak{g}$. Then we have*

$$C^\infty_{\gamma_0}(S^2T^*_{\mathbb{C}}) = \{\, h \in C^\infty(S^2T^*_{\mathbb{C}}) \mid \Delta h = 2\lambda h \,\},$$
$$E(X) = \{\, h \in C^\infty_{\gamma_0}(S^2_0T^*_{\mathbb{C}}) \mid h = \bar{h},\ \operatorname{div} h = 0 \,\}.$$

Using this proposition, in [13] and [14] Koiso determines all the irreducible symmetric spaces of compact type whose infinitesimal Einstein deformations vanish. In particular, the space $E(X)$ vanishes when $X = G^{\mathbb{R}}_{m,n}$, with $(m,n) \neq (3,3)$, or when $X = G^{\mathbb{C}}_{1,n}$, with $n \geq 2$. On the other hand, the space $E(X)$ is non-zero when $X = G^{\mathbb{R}}_{3,3}$, or when $X = G^{\mathbb{C}}_{m,n}$, with $m, n \geq 2$.

5. A criterion for Guillemin rigidity

We shall now give a criterion for the Guillemin rigidity of X which exploits

(i) the fact that X is an Einstein manifold;

(ii) the hereditary properties of the operator D_1 with respect to totally geodesic submanifolds;

(iii) the previously known results about Guillemin rigidity.

We choose a family $\mathcal{F}'$ of closed connected totally geodesic submanifolds of X which are known to be rigid in the sense of Guillemin and a family $\mathcal{F}$ of closed connected totally geodesic surfaces of X each of which is contained in a submanifold belonging to $\mathcal{F}'$. Assume that the family $\mathcal{F}$ is invariant under the group G.

The set N consisting of those elements of B, which vanish when restricted to the closed totally geodesic submanifolds of $\mathcal{F}$, is a sub-bundle of B. In fact, the infinitesimal orbit of the curvature $\tilde{B}$ is a sub-bundle of N, and we identify $N/\tilde{B}$ with a sub-bundle of $B/\tilde{B}$.

We denote by $\mathcal{L}(\mathcal{F}')$ the subspace of $C^\infty(S^2T^*)$ consisting of all symmetric 2-forms h satisfying the following condition: for all submanifolds $Z \in \mathcal{F}'$, the restriction of h to Z is a Lie derivative of the metric of Z induced by g.

Using the vanishing of the infinitesimal orbits of the submanifolds belonging to the family $\mathcal{F}$, we obtain:

Proposition 4. *A symmetric* 2*-form h on X belonging to $\mathcal{L}(\mathcal{F}')$ satisfies the relation $D_1 h \in C^\infty(N/\tilde{B})$.*

By means of Lemma 1 and Proposition 4, we obtain a criterion for the Guillemin rigidity of the irreducible symmetric space X of compact type, which may be formulated as follows:

Theorem 7. *Let E be a G-sub-bundle of $S_0^2T^*$. Assume that the relations*

$$\operatorname{Tr} N \subset E, \qquad C^\infty(E) \cap \mathcal{L}(\mathcal{F}') = \{0\}, \qquad \mathcal{N} \cap E(X) = \{0\} \tag{4}$$

hold. Suppose that, whenever a section of S^2T^* over X satisfies the Guillemin condition, its restriction to an arbitrary submanifold of X belonging to the family $\mathcal{F}'$ satisfies the Guillemin condition. Then the symmetric space X is rigid in the sense of Guillemin.

6. Rigidity of the real and complex Grassmannians

Let X be the real Grassmannian $G^{\mathbb{R}}_{m,n}$, with $m, n \geq 3$ and $m \neq n$, which is an irreducible symmetric space. Let V be the canonical vector bundle (of rank m) whose fiber at $x \in X$ is the subspace of $\mathbb{R}^{m+n}$ determined by the m-plane x and let W be the vector bundle of rank n over X whose fiber at $x \in X$ is the orthogonal complement W_x of V_x in $\mathbb{R}^{m+n}$. In this case, the group G is equal to $SO(m+n)$.

The tangent bundle T of X is canonically isomorphic to the vector bundle $\operatorname{Hom}(V, W)$ and so we may identify it with $V \otimes W$. We have the equality

$$S^2T^* = (S^2V^* \otimes S^2W^*) \oplus (\textstyle\bigwedge^2 V^* \otimes \bigwedge^2 W^*).$$

The sub-bundle $\bigwedge^2 V^* \otimes \bigwedge^2 W^*$ of S^2T^* can be identified with the G-invariant sub-bundle E consisting of all elements h of S^2T^* satisfying

$$h(\xi, \xi) = 0,$$

for all elements ξ of $V \otimes W$ of rank one.

Let $\mathcal{F}'$ be the family consisting of totally geodesic submanifolds of X isometric to the Grassmannian $G^{\mathbb{R}}_{2,n}$. Let $\mathcal{F}$ be the family consisting of totally geodesic surfaces of X which are contained in some member of the family $\mathcal{F}'$ and which are either isometric to a flat 2-torus or to a 2-sphere of constant curvature 1. According to Koiso [13] and [14], the space $E(X)$ vanishes. To prove the rigidity of the Grassmannian X, we shall apply Theorem 7 to X, the families $\mathcal{F}$ and $\mathcal{F}'$ and this sub-bundle E of S^2T^*.

Using the injectivity the Radon transform on the real projective plane $\mathbb{RP}^2$, we show that condition (i) of Theorem 7 holds. The second equality of (4) is a consequence of the following theorem which is proved by the same methods used to demonstrate Theorem 5.

THEOREM 8. *A section of the vector bundle E over the Grassmannian $X = G^{\mathbb{R}}_{2,3}$ satisfying the Guillemin condition vanishes.*

Finally, we consider the complex Grassmannian $X = G^{\mathbb{C}}_{m,n}$, with $m, n \geq 2$ and $m \neq n$. In this case, the group G is equal to $SU(m+n)$ and X is not equal to a simple Lie group. Here we encounter an additional difficulty arising from the fact that $E(X)$ is non-zero. By Proposition 3, we know that $E(X)$ is a subspace of the G-module $C^{\infty}_{\gamma_0}(S^2_0 T^*_{\mathbb{C}})$. To show that the last equality of (4) holds, we find explicit formulas for the highest weight vectors of the G-module $C^{\infty}_{\gamma_0}(S^2_0 T^*_{\mathbb{C}})$, and then carry out integrations of these tensors over certain closed geodesics in order to prove the following stronger result:

PROPOSITION 5. *Let X be the complex Grassmannian $X = G^{\mathbb{C}}_{m,n}$, with $m, n \geq 2$ and $m \neq n$, and let h be an element of $C^{\infty}_{\gamma_0}(S^2_0 T^*_{\mathbb{C}})$. If h satisfies the Guillemin condition, then h vanishes.*

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